Analyse vectorielle

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Opérateur gradient

Cartésien

$$\overrightarrow{\text{grad}} f(M) = \left(\frac{\partial f}{\partial x}\right)\overrightarrow{e_x} + \left(\frac{\partial f}{\partial y}\right)\overrightarrow{e_y} + \left(\frac{\partial f}{\partial z}\right)\overrightarrow{e_z}$$

Cylindrique

$$\overrightarrow{\text{grad}} f(M) = \left(\frac{\partial f}{\partial r}\right)\overrightarrow{e_r} + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\overrightarrow{e_\theta} + \left(\frac{\partial f}{\partial z}\right)\overrightarrow{e_z}$$

Sphérique

$$\overrightarrow{\text{grad}} f(M) = \left(\frac{\partial f}{\partial r}\right)\overrightarrow{e_r} + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\overrightarrow{e_\theta} + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial \varphi}\right)\overrightarrow{e_\varphi}$$

Opérateur divergence

Cartésien

$$\text{div}\overrightarrow{F}(x,y,z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Cylindrique

$$\text{div}\overrightarrow{F}(r,\theta,z) = \frac{1}{r}\frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Sphérique

$$\text{div}\overrightarrow{F}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (F_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

Opérateur rotationnel

Cartésien

$$\overrightarrow{\text{rot}}\overrightarrow{F}(x,y,z) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\overrightarrow{e_x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\overrightarrow{e_y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\overrightarrow{e_z}$$

Cylindrique

$$\overrightarrow{\text{rot}}\overrightarrow{F}(r,\theta,z) = \frac{1}{r}\left(\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\overrightarrow{e_r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\overrightarrow{e_\theta} + \frac{1}{r}\left(\frac{\partial F_\theta}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\overrightarrow{e_z}$$

Sphérique

$$\overrightarrow{\text{rot}}\overrightarrow{F}(r,\theta,\varphi) = \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta F_\varphi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \varphi}\right)\overrightarrow{e_r} + \frac{1}{r\sin\theta}\left(\frac{\partial F_r}{\partial \varphi} - \frac{\partial (r\sin\theta F_\varphi)}{\partial r}\right)\overrightarrow{e_\theta} + \frac{1}{r}\left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\overrightarrow{e_\varphi}$$

Opérateur Laplacien Scalaire

Cartésien

$$\Delta f(x,y,z) = \left(\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial^2 f}{\partial z^2}\right)$$

Cylindrique

$$\Delta f(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Sphérique

$$\Delta f(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Opérateur Laplacien Vectoriel

Définition

$$\overrightarrow{\Delta}\overrightarrow{F} = \overrightarrow{\mathrm{grad}}\,\mathrm{div}\overrightarrow{F} - \overrightarrow{\mathrm{rot}}\,\overrightarrow{\mathrm{rot}}\overrightarrow{F}$$

Cartésien

$$\overrightarrow{\Delta}\overrightarrow{F} = \Delta F_x\overrightarrow{e_x} + \Delta F_y\overrightarrow{e_y} + \Delta F_z\overrightarrow{e_z}$$

Produits de champs

$$\overrightarrow{\mathrm{grad}}(UV) = V\overrightarrow{\mathrm{grad}}(U) + U\overrightarrow{\mathrm{grad}}(V)$$

$$\overrightarrow{\mathrm{rot}}(U\overrightarrow{A}) = \overrightarrow{\mathrm{grad}}(U) \wedge \overrightarrow{A} + U\overrightarrow{\mathrm{rot}}(\overrightarrow{A})$$

$$\mathrm{div}(U\overrightarrow{A}) = \overrightarrow{A}\overrightarrow{\mathrm{grad}}(U) + U\mathrm{div}(\overrightarrow{A})$$

$$\mathrm{div}(\overrightarrow{A} \wedge \overrightarrow{B}) = \overrightarrow{B} \cdot \overrightarrow{\mathrm{rot}}(\overrightarrow{A}) - \overrightarrow{A} \cdot \overrightarrow{\mathrm{rot}}(\overrightarrow{B})$$

Combinaisons d'opérateurs

$$\mathrm{div}\,(\overrightarrow{\mathrm{grad}}U) = \Delta U$$

$$\overrightarrow{\mathrm{rot}}\,(\overrightarrow{\mathrm{grad}}\overrightarrow{A}) = \overrightarrow{0}$$

$$\mathrm{div}\,(\overrightarrow{\mathrm{rot}}\overrightarrow{A}) = 0$$

$$\overrightarrow{\Delta}\overrightarrow{F} = \overrightarrow{\mathrm{grad}}\,\mathrm{div}\overrightarrow{F} - \overrightarrow{\mathrm{rot}}\,\overrightarrow{\mathrm{rot}}\overrightarrow{F}$$

Théorème de Green-Ostrogradski

$$\oiint_\Sigma \overrightarrow{F}(M)\mathrm{d}\overrightarrow{S}_{\text{sortant}} = \iiint_V \mathrm{div}\overrightarrow{F}.\mathrm{d}V$$

Théorème de Stokes-Ampère

$$\oint_\Gamma \overrightarrow{F}(M)\mathrm{d}\overrightarrow{\ell} = \iint_\Sigma \overrightarrow{\mathrm{rot}}\,\overrightarrow{F} \cdot \mathrm{d}\overrightarrow{S}$$